A Closed Form Solution of a Crack in Magneto-Electro-Elastic Composites under Anti-Plane Shear Stress Loading*

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In this paper, a closed form solution of a crack in magneto-electro-elastic composites under anti-plane shear stress loading is obtained for the permeable crack surface conditions. By using the Fourier transform technique, the problem can be solved with a pair of dual integral equations in which the unknown variable is the jump of the displacements across the crack surfaces. In solving the dual integral equations, the jump of the displacements across the crack surface is expanded in a series of Jacobi polynomials. The closed form solutions of the stress intensity factor, the electric displacement intensity factor and the magnetic flux intensity factor are given. It can be obtained that the stress field is independent of the electric field and the magnetic flux.

***Key Words*:** Magneto-Electro-Elastic Composites, Crack, Dual Integral Equations, Intensity Factor

1. Introduction

Combining two or more distinct piezoelectric and piezomagnetic (magnetostrictive) constituents, piezoelectric/piezoemagnetic composite materials can take the advantages of each constituent and consequently have superior coupling magnetoelectric effect as compared to conventional piezoelectric or piezomagnetic materials. The magnetoelectric coupling is a new product property of the composite, since it is absent in each constituent. In some cases, the coupling effect of piezoelectric/piezomagnetic composites can be even obtained a hundred times larger than that in a single-phase magnetoelectric material. Consequently, they are extensively used as magnetic field probes, electric packaging, acoustic, hydrophones, medical ultrasonic imaging, sensors, and actuators with the responsibility of magneto-electro-mechanical energy conversion(1). When subjected to mechanical, magnetic and electrical loads in service, these magneto-electro-elastic composites can fail prematurely due to some defects, e.g. cracks, holes, etc. arising during their manufacturing process. Therefore, it is of great importance to study the magneto-electro-elastic interaction and fracture behavior of magneto-electro-elastic composites(2),(3).

The development of piezoelectric-piezomagnetic composites has its roots in the early work of Van Suchtelen(4) who proposed that the combination of piezoelectric-piezomagnetic phases may exhibit a new material property — the magnetoelectric coupling effect. Since then, the magnetoelectric coupling effect of $BaTiO_3$-$CoFe_2O_4$ composites has been measured by many researchers. Much of the theoretical work for the investigation of magnetoelectric coupling effect has only recently been studied(1)−(3),(5)−(10). It appears that these approaches have not provided a means to find a closed form solution of a crack in magneto-electro-elastic composites under anti-plane shear stress loading. Thus, the present work is an attempt to fill this information needed. The solving process is quite different from those adopted in the Refs. (2) and (3).

2. Formulation of the Problem

It is assumed that there is a crack of length $2l$ in magneto-electro-elastic composites as shown in Fig. 1. The piezoelectric/piezomagnetic boundary-value problem for anti-plane shear is considerably simplified if we consider only the out-of-plane displacement, the in-plane electric and the in-plane magnetic fields. As discussed in Soh's work(11), since no opening displacement exists for the present anti-plane problem, the crack surfaces can be assumed to be in perfect contact. Accordingly, permeable condition will be enforced in the present study, i.e., the electric potential, the normal electric displacement, the

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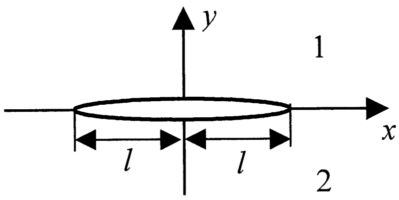

Fig. 1 A crack in magneto-electro-elastic composites

magnetic flux and the magnetic potential are assumed to be continuous across the crack sufaces. So the boundary conditions of the present problem are:

$$\begin{cases} \tau_{yz}^{(1)}(x,0^+) = \tau_{yz}^{(2)}(x,0^-) = -\tau_0, & |x| \le l \\ w^{(1)}(x,0^+) = w^{(2)}(x,0^-), & |x| > l \end{cases} \tag{1}$$

$$\begin{cases} \tau_{yz}^{(1)}(x,0^+) = \tau_{yz}^{(2)}(x,0^-) \\ \phi^{(1)}(x,0^+) = \phi^{(2)}(x,0^-), \quad |x| \le \infty \\ D_y^{(1)}(x,0^+) = D_y^{(2)}(x,0^-) \end{cases} \tag{2}$$

$$\psi^{(1)}(x,0^+) = \psi^{(2)}(x,0^-), \quad B_y^{(1)}(x,0^+) = B_y^{(2)}(x,0^-), \quad |x| \le \infty \tag{3}$$

$$w^{(1)}(x,y) = w^{(2)}(x,y) = 0 \quad \text{for} \quad (x^2+y^2)^{1/2} \to \infty \tag{4}$$

where $\tau_{zk}^{(i)}$, $D_k^{(i)}$ and $B_k^{(i)}$ ($k = x, y, i = 1, 2$) are the anti-plane shear stress, in-plane electric displacement and in-plane magnetic flux, respectively. $w^{(i)}$, $\phi^{(i)}$ and $\psi^{(i)}$ are the mechanical displacement, the electric potential and the magnetic potential. Note that all quantities with superscript i (i=1, 2) refer to the upper half plane 1 and the lower half plane 2 as in Fig. 1, respectively. In this paper, we only consider that τ_0 is positive.

The constitutive equations can be written as

$$\tau_{zk}^{(i)} = c_{44} w_{,k}^{(i)} + e_{15}\phi_{,k}^{(i)} + q_{15}\psi_{,k}^{(i)}, \quad (k = x, y, i = 1, 2) \tag{5}$$

$$D_k^{(i)} = e_{15} w_{,k}^{(i)} - \varepsilon_{11}\phi_{,k}^{(i)} - d_{11}\psi_{,k}^{(i)}, \quad (k = x, y, i = 1, 2) \tag{6}$$

$$B_k^{(i)} = q_{15} w_{,k}^{(i)} - d_{11}\phi_{,k}^{(i)} - \mu_{11}\psi_{,k}^{(i)}, \quad (k = x, y, i = 1, 2) \tag{7}$$

where c_{44} is shear modulus, e_{15} is piezoelectric coefficient, ε_{11} is dielectric parameter, q_{15} is piezomagnetic coefficient, d_{11} is magnetioelectric coefficient, μ_{11} is magnetic permeability.

The anti-plane governing equations are

$$c_{44}\nabla^2 w^{(i)} + e_{15}\nabla^2\phi^{(i)} + q_{15}\nabla^2\psi^{(i)} = 0, \quad (i = 1, 2) \tag{8}$$

$$e_{15}\nabla^2 w^{(i)} - \varepsilon_{11}\nabla^2\phi^{(i)} - d_{11}\nabla^2\psi^{(i)} = 0, \quad (i = 1, 2) \tag{9}$$

$$q_{15}\nabla^2 w^{(i)} - d_{11}\nabla^2\phi^{(i)} - \mu_{11}\nabla^2\psi^{(i)} = 0, \quad (i = 1, 2) \tag{10}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator. Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for $0 \le x < \infty$, $-\infty \le y < \infty$ only. A Fourier transform is applied to Eqs. (8) – (10). Assume that the solutions are

$$\begin{cases} w^{(1)}(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^\infty A_1(s)e^{-sy}\cos(sx)ds \\ \phi^{(1)}(x,y) = \dfrac{\mu_{11}e_{15} - d_{11}q_{15}}{\varepsilon_{11}\mu_{11} - d_{11}^2} w^{(1)}(x,y) \\ \quad + \dfrac{2}{\pi}\displaystyle\int_0^\infty B_1(s)e^{-sy}\cos(sx)ds, \quad (y \ge 0) \\ \psi^{(1)}(x,y) = \dfrac{q_{15}\varepsilon_{11} - d_{11}e_{15}}{\varepsilon_{11}\mu_{11} - d_{11}^2} w^{(1)}(x,y) \\ \quad + \dfrac{2}{\pi}\displaystyle\int_0^\infty C_1(s)e^{-sy}\cos(sx)ds \end{cases} \tag{11}$$

$$\begin{cases} w^{(2)}(x,y) = \dfrac{2}{\pi}\displaystyle\int_0^\infty A_2(s)e^{sy}\cos(sx)ds \\ \phi^{(2)}(x,y) = \dfrac{\mu_{11}e_{15} - d_{11}q_{15}}{\varepsilon_{11}\mu_{11} - d_{11}^2} w^{(2)}(x,y) \\ \quad + \dfrac{2}{\pi}\displaystyle\int_0^\infty B_2(s)e^{sy}\cos(sx)ds, \quad (y \le 0) \\ \psi^{(2)}(x,y) = \dfrac{q_{15}\varepsilon_{11} - d_{11}e_{15}}{\varepsilon_{11}\mu_{11} - d_{11}^2} w^{(2)}(x,y) \\ \quad + \dfrac{2}{\pi}\displaystyle\int_0^\infty C_2(s)e^{sy}\cos(sx)ds \end{cases} \tag{12}$$

where $A_1(s)$, $B_1(s)$, $C_1(s)$, $A_2(s)$, $B_2(s)$ and $C_2(s)$ are unknown functions.

So from Eqs. (5) – (7), we have

$$\tau_{yz}^{(1)}(x,y) = -\frac{2}{\pi}\int_0^\infty s\left[\left(c_{44} + \frac{a_1 e_{15}}{a_0} + \frac{a_2 q_{15}}{a_0}\right)A_1(s) + e_{15}B_1(s) + q_{15}C_1(s)\right]e^{-sy}\cos(sx)ds \tag{13}$$

$$D_y^{(1)}(x,y) = \frac{2}{\pi}\int_0^\infty s[\varepsilon_{11}B_1(s) + d_{11}C_1(s)]e^{-sy}\cos(sx)ds \tag{14}$$

$$B_y^{(1)}(x,y) = \frac{2}{\pi}\int_0^\infty s[d_{11}B_1(s) + \mu_{11}C_1(s)]e^{-sy}\cos(sx)ds \tag{15}$$

$$\tau_{yz}^{(2)}(x,y) = \frac{2}{\pi}\int_0^\infty s\left[\left(c_{44} + \frac{a_1 e_{15}}{a_0} + \frac{a_2 q_{15}}{a_0}\right)A_2(s) + e_{15}B_2(s) + q_{15}C_2(s)\right]e^{sy}\cos(sx)ds \tag{16}$$

$$D_y^{(2)}(x,y) = -\frac{2}{\pi}\int_0^\infty s[\varepsilon_{11}B_2(s) + d_{11}C_2(s)]e^{sy}\cos(sx)ds \tag{17}$$

$$B_y^{(2)}(x,y) = -\frac{2}{\pi}\int_0^\infty s[d_{11}B_2(s) + \mu_{11}C_2(s)]e^{sy}\cos(sx)ds \tag{18}$$

where $a_0 = \varepsilon_{11}\mu_{11} - d_{11}^2$, $a_1 = \mu_{11}e_{15} - d_{11}q_{15}$, $a_2 = q_{15}\varepsilon_{11} - d_{11}e_{15}$.

For solving the problem, the jumps of the displacements, the electric and the magnetic potentials across the crack surfaces are defined as follows:

$$f(x) = w^{(1)}(x,0^+) - w^{(2)}(x,0^-) \tag{19}$$

$$\begin{cases} f_\phi(x) = \phi^{(1)}(x,0^+) - \phi^{(2)}(x,0^-) \\ f_\psi(x) = \psi^{(1)}(x,0^+) - \psi^{(2)}(x,0^-) \end{cases} \tag{20}$$

Substituting Eqs. (11) and (12) into Eqs. (19) – (20), and applying the Fourier transform and the boundary conditions, it can be obtained

$$\bar{f}(s) = A_1(s) - A_2(s) \tag{21}$$

$$\frac{a_1}{a_0}[A_1(s) - A_2(s)] + B_1(s) - B_2(s) = 0 \tag{22}$$

$$\frac{a_2}{a_0}[A_1(s) - A_2(s)] + C_1(s) - C_2(s) = 0 \tag{23}$$

Substituting Eqs. (13) – (18) into Eqs. (1) – (3), it can be obtained

$$\left(c_{44} + \frac{a_1 e_{15}}{a_0} + \frac{a_2 q_{15}}{a_0}\right) A_1(s) + e_{15} B_1(s) + q_{15} C_1(s) + \left(c_{44} + \frac{a_1 e_{15}}{a_0} + \frac{a_2 q_{15}}{a_0}\right) A_2(s) + e_{15} B_2(s) + q_{15} C_2(s) = 0 \tag{24}$$

$$\varepsilon_{11} B_1(s) + d_{11} C_1(s) + \varepsilon_{11} B_2(s) + d_{11} C_2(s) = 0 \tag{25}$$

$$d_{11} B_1(s) + \mu_{11} C_1(s) + d_{11} B_2(s) + \mu_{11} C_2(s) = 0 \tag{26}$$

By solving six equations (21) – (26) with six unknown functions $A_1(s)$, $B_1(s)$, $C_1(s)$, $A_2(s)$, $B_2(s)$, $C_2(s)$ and applying the boundary condition (1), it can be obtained:

$$\frac{c_{44}}{\pi} \int_0^\infty s \bar{f}(s) \cos(sx) ds = \tau_0, \quad |x| \le l \tag{27}$$

$$\int_0^\infty \bar{f}(s) \cos(sx) ds = 0, \quad |x| > l \tag{28}$$

To determine the unknown function $\bar{f}(s)$, the dual-integral equations (27) and (28) must be solved.

3. Solution of the Dual-Integral Equations

To solve the dual-integral equations (27) and (28), the jump of the displacements across the crack surfaces is represented by the following series:

$$f(x) = \sum_{n=1}^{\infty} b_n P_{2n-2}^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{\frac{1}{2}}, \quad \text{for} \quad -l \le x \le l \tag{29}$$

where b_n is unknown coefficients to be determined and $P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x)$ is a Jacobi polynomial[12]. The Fourier transform of Eq. (29) are[13]

$$\bar{f}(s) = \sum_{n=1}^{\infty} b_n G_n \frac{1}{s} J_{2n-1}(sl), \; G_n = 2\sqrt{\pi}(-1)^{n-1} \frac{\Gamma\left(2n - \frac{1}{2}\right)}{(2n-2)!} \tag{30}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Eq. (30) into Eqs. (27) and (28), Eq. (28) has been automatically satisfied. After integration with respect to x in $[0, x]$, Eq. (27) reduces to

$$\sum_{n=1}^{\infty} b_n G_n \int_0^\infty \frac{1}{s} J_{2n-1}(sl) \sin(sx) ds = \frac{\pi \tau_0 x}{c_{44}} \tag{31}$$

From the relationship[12]

$$\int_0^\infty \frac{1}{s} J_n(sa) \sin(bs) ds = \begin{cases} \dfrac{\sin[n \sin^{-1}(b/a)]}{n}, & a > b \\ \dfrac{a^n \sin(n\pi/2)}{n[b + \sqrt{b^2 - a^2}]^n}, & b > a \end{cases}$$

it can be obtained that

$$b_1 = \frac{\pi \tau_0 l}{G_1 c_{44}} = \frac{\tau_0 l}{c_{44}}, \quad b_n = 0, \quad n = 2, 3, 4, \ldots \tag{32}$$

4. Intensity Factors

The coefficients b_n are known, so that the entire perturbation stress field, the perturbation electric displacement and the magnetic flux can be obtained. However, in fracture mechanics, it is of importance to determine the perturbation stress $\tau_{yz}^{(1)}$, the perturbation electric displacement $D_y^{(1)}$ and the magnetic flux $B_y^{(1)}$ in the vicinity of the crack's tips. In the case of the present study, $\tau_{yz}^{(1)}$, $D_y^{(1)}$ and $B_y^{(1)}$ along the crack line can be expressed respectively as

$$\tau_{yz}^{(1)}(x, 0) = -\frac{c_{44}}{\pi} \sum_{n=1}^{\infty} b_n G_n \int_0^\infty J_{2n-1}(sl) \cos(xs) ds = -\tau_0 l \int_0^\infty J_1(sl) \cos(xs) ds \tag{33}$$

$$D_y^{(1)}(x, 0) = -\frac{e_{15}}{\pi} \sum_{n=1}^{\infty} b_n G_n \int_0^\infty J_{2n-1}(sl) \cos(xs) ds = -\frac{e_{15} \tau_0 l}{c_{44}} \int_0^\infty J_1(sl) \cos(xs) ds \tag{34}$$

$$B_y^{(1)}(x, 0) = -\frac{q_{15}}{\pi} \sum_{n=1}^{\infty} b_n G_n \int_0^\infty J_{2n-1}(sl) \cos(xs) ds = -\frac{q_{15} \tau_0 l}{c_{44}} \int_0^\infty J_1(sl) \cos(xs) ds \tag{35}$$

The singular parts of the stress field, the electric displacement and the magnetic flux in Eqs. (33) – (35) can be obtained respectively from the relationship[12]

$$\int_0^\infty J_n(sa) \cos(bs) ds = \begin{cases} \dfrac{\cos[n \sin^{-1}(b/a)]}{\sqrt{a^2 - b^2}}, & a > b \\ -\dfrac{a^n \sin(n\pi/2)}{\sqrt{b^2 - a^2}[b + \sqrt{b^2 - a^2}]^n}, & b > a \end{cases} \tag{36}$$

The singular parts of the stress field, the electric displacement and the magnetic flux can be expressed respectively as follows ($x > l$):

$$\tau = \tau_0 l H(x) \tag{37}$$

$$D = \frac{e_{15} \tau_0 l}{c_{44}} H(x) \tag{38}$$

$$B = \frac{q_{15} \tau_0 l}{c_{44}} H(x) \tag{39}$$

where $H(x) = \dfrac{l}{\sqrt{x^2 - l^2}[x + \sqrt{x^2 - l^2}]}$

We obtain the stress intensity factor K as

$$K = \lim_{x \to l^+} \sqrt{2(x - l)} \cdot \tau = \tau_0 \sqrt{l} \tag{40}$$

We obtain the electric displacement intensity factor D_L as

$$D_L = \lim_{x \to l^+} \sqrt{2(x-l)} \cdot D = \frac{e_{15}}{c_{44}} K \tag{41}$$

We obtain the magnetic flux intensity factor B_L as

$$B_L = \lim_{x \to l^+} \sqrt{2(x-l)} \cdot B = \frac{q_{15}}{c_{44}} K \tag{42}$$

5. Conclusions

From the results, the following observations are very significant:

(i) The solution of the present problem is a closed form. The form of the stress intensity factor of the present paper is the same as the one in a general elastic material.

(ii) The stress intensity factor does not depend on the material constants. However, the electric displacement intensity factor depends on the shear modulus and the dielectric parameter, the magnetic flux intensity factor depends on the shear modulus and the piezomagnetic coefficient as shown in Eqs. (40) – (42).

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